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The Laplacian spectral radius of graphs on surfaces

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Abstract

Let G be an n-vertex ($n \geqslant 3$) simple graph embeddable on a surface of Euler genus γ (the number of crosscaps plus twice the number of handles). Denote by Δ the maximum degree of G. In this paper, we first present two upper bounds on the Laplacian spectral radius of G as follows:

(i)

$$\lambda_1(G) \leqslant \frac{\Delta + 4 + \sqrt{(\Delta + 4)^2 + 8(2n + 8\gamma - 10)}}{2}.$$

(ii) If G is 4-connected and either the surface is the sphere or the embedding is 4-representative, then

$$\lambda_1(G) \leqslant \frac{\Delta + 2 + \sqrt{(\Delta + 2)^2 + 8(2n + 2\gamma - 4)}}{2}.$$

Some upper bounds on the Laplacian spectral radius of the outerplanar and Halin graphs are also given.

Keywords: Adjacency matrix; Laplacian matrix; Spectral radius; Euler genus; Outerplanar graph; Halin graph

1. Introduction

Let $G = (V, E)$ be a simple undirected graph with n vertices. For $v \in V$, the degree of v, written by $d(v)$, is the number of edges incident with v. Denote by Δ the maximum degree of G.

Let $A(G)$ be the adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees. Then the Laplacian matrix of G is $L(G) = D(G) - A(G)$. Let $Q(G) = D(G) + A(G)$. Since $Q(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are real numbers. For a matrix

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M, we denote by $\lambda_1(M)$ the largest eigenvalue of M, while for a graph G, we will use $\lambda_1(G)$ to denote $\lambda_1(L(G))$ and call it the Laplacian spectral radius of G.

Let Σ be a compact surface and γ be the Euler genus (the number of crosscaps plus twice the number of handles) of Σ. An embedding is k-representative if no noncontractible closed curve in the surface intersects the embedded graph at fewer than k points. An embedding is cellular if every face is homeomorphic to an open disk.

In particular, if $\gamma = 0$, Σ is a plane. We call a graph G a *planar* graph, if G can be embedded in a plane such that no two edges intersect. Now we give the definitions of the two kinds of special planar graph. A graph is *outerplanar* if it has a planar embedding in which every vertex lies on the outer face. An outerplanar graph G is *maximal* if for every pair of nonadjacent vertices u and v of G, the graph $G + uv$ is nonouterplanar. Outerplanar graphs have been widely studied since they have many applications [3] and interesting theoretical properties [2,5,6].

Let T be a tree with $n \geqslant 4$ vertices and without vertices of degree 2. If T is embedded in the plane with its end-vertices $v_1, v_2, \dots, v_t$ under the rotation of T and new edges (v_i, v_{i+1}) (where $v_{t+1} = v_1$) are added to the edge set of T, then T together with the cycle $(v_1, v_2, \dots, v_t)$ forms a 3-connected planar graph G called *Halin graph*. Vertices v_i $(1 \leqslant i \leqslant t)$ is called an *outer vertex* and any other vertex is called *inner vertex*. Denote $IV(G) = \{v\colon v \text{ is an inner vertex of } V(G)\}$ and $a = |IV(G)|$.

In this paper, we first present two new upper bounds on the Laplacian spectral radius of the graph embeddable on Σ in terms of n, Δ and γ. Then we give some upper bounds on the Laplacian spectral radius of the maximal outerplanar and Halin graphs.

2. Lemmas and results

First, we give some lemmas which will be used in our proof.

Lemma 1 [8]. *Let G be a graph. Then*

$$\lambda_1(G) \leqslant \lambda_1(Q).$$

Moreover, if G is connected, then the equality holds if and only if G is a bipartite graph.

Let B be a matrix. Denote the ith row sum of B by $s_v(B)$.

Lemma 2 [4]. *Let G be an n-vertex graph, $Q = Q(G)$ and P any polynomial. Then*

$$\min_{v\in V(G)} s_v(P(Q)) \leqslant \lambda_1(P(Q)) \leqslant \max_{v\in V(G)} s_v(P(Q)).$$

Let G be a graph and $v \in V(G)$. Denote by $N_i(v, G)$ the set of vertices at distance i from v and let $n_i(v, G) = |N_i(v, G)|$.

Lemma 3 [1]. *Let G be a graph on at least two vertices, with adjacency matrix A and with a cellular embedding Ψ in a surface of Euler genus γ. Let $v \in V(G)$ and $c(v, \Psi)$ be the number of edges that join two vertices of $N_1(v, G)$ that are not consecutive in the embedded order around v. If $n_1(v, G) \geqslant 3$, then we have*

(i) $s_v(A^2) \leqslant 4n_1(v, G) + 2n_2(v, G) + 2c(v, \Psi) + 2\gamma - 2$,
(ii) $s_v(A^2) \leqslant 6n_1(v, G) + 2n_2(v, G) + 8\gamma - 8$.

Lemma 4 [7]. *Let G be a maximal outerplanar graph of order n ($n \geqslant 2$) and $v \in V(G)$. Then*

$$s_v(A^2 - 3A) \leqslant n - 4.$$

Lemma 5 [7]. *Let G be a Halin graph of order n and $v \in V(G)$. Then*

(i) $s_v(A^2 - 2A) \leqslant n - 2a + 1$;
(ii) $\Delta(G) \leqslant n - 2a + 1$.

Our main result of the paper is the following theorem.

Theorem 6. *Let G be an n-vertex graph, $n \geqslant 3$, with maximum degree Δ. Suppose G can be embedded on a surface of Euler genus γ.*

(i) *Then*

$$\lambda_1(G) \leqslant \frac{\Delta + 4 + \sqrt{(\Delta+4)^2 + 8(2n + 8\gamma - 10)}}{2}.$$

(ii) *If G is 4-connected and either the surface is the sphere or the embedding is 4-representative, then*

$$\lambda_1(G) \leqslant \frac{\Delta + 2 + \sqrt{(\Delta+2)^2 + 8(2n + 2\gamma - 4)}}{2}.$$

Proof. We can assume that the embedding is celluar and $n_1(v, G) \geqslant 3$ (see [1, Theorem 3.1]).

Since $Q = D + A$, we have $s_v(Q) = 2d(v)$. Note that $s_v(AD) = s_v(A^2)$. Then

$$\begin{aligned} s_v(Q^2) &= s_v(D(D + A) + AD + A^2) \\ &= d(v)s_v(Q) + s_v(AD) + s_v(A^2) \\ &\leqslant \Delta s_v(Q) + 2s_v(A^2). \end{aligned} \tag{1}$$

By Lemma 3 (ii),

$$s_v(Q^2) \leqslant \Delta s_v(Q) + 2(6n_1 + 2n_2 + 8\gamma - 8).$$

Since $n \geqslant n_1 + n_2 + 1$, we have

$$s_v(Q^2) \leqslant \Delta s_v(Q) + 2(2n + 8\gamma - 10) + 8n_1.$$

Note that $s_v(Q) = 2d(v) = 2n_1$. Thus

$$s_v(Q^2) - (\Delta + 4)s_v(Q) \leqslant 2(2n + 8\gamma - 10).$$

By Lemma 2, we have

$$\lambda_1^2(Q) - (\Delta + 4)\lambda_1(Q) \leqslant 2(2n + 8\gamma - 10).$$

Solving the quadratic inequality, we obtain

$$\lambda_1(Q) \leqslant \frac{\Delta + 4 + \sqrt{(\Delta+4)^2 + 8(2n + 8\gamma - 10)}}{2}.$$

By Lemma 1

$$\lambda_1(G) \leqslant \frac{\Delta + 4 + \sqrt{(\Delta+4)^2 + 8(2n + 8\gamma - 10)}}{2}.$$

If G is connected, and if the embedding Ψ of G is 4-representative when it is not on the sphere, then for each vertex v we have $n_1(v, G) \geqslant 4$ and $c(v, \Psi) = 0$ (see [1, Theorem 3.1]). Hence by inequality (1) and Lemma 3 (i), we have that

$$s_v(Q^2) \leqslant \Delta s_v(Q) + 2s_v(A^2) \leqslant \Delta s_v(Q) + 2(4n_1 + 2n_2 + 2\gamma - 2).$$

By the similar argument as the above, we obtain

$$\lambda_1(Q) \leqslant \frac{\Delta + 2 + \sqrt{(\Delta + 2)^2 + 8(2n + 2\gamma - 4)}}{2}.$$

Hence, by Lemma 1

$$\lambda_1(G) \leqslant \frac{\Delta + 2 + \sqrt{(\Delta + 2)^2 + 8(2n + 2\gamma - 4)}}{2}. \qquad \square$$

Corollary 7. *Let G be an n-vertex planar graph, $n \geqslant 3$*

(i) *Then*

$$\lambda_1(G) \leqslant \frac{\Delta + 4 + \sqrt{(\Delta + 4)^2 + 8(2n - 10)}}{2}.$$

(ii) *If G is 4-connected, then*

$$\lambda_1(G) \leqslant \frac{\Delta + 2 + \sqrt{(\Delta + 2)^2 + 8(2n - 4)}}{2}.$$

Theorem 8. *Let G be an n-vertex graph, $n \geqslant 3$, with maximum degree Δ*

(i) *If G is a maximal outerplanar graph, then*

$$\lambda_1(G) \leqslant \frac{\Delta + 3 + \sqrt{(\Delta + 3)^2 + 8(n - 4)}}{2}. \tag{2}$$

(ii) *If G is a Halin graph and $a = |IV(G)|$, then*

$$\lambda_1(G) \leqslant \frac{\Delta + 2 + \sqrt{(\Delta + 2)^2 + 8(n - 2a + 1)}}{2}. \tag{3}$$

Proof. If G is a maximal outerplanar graph, by Lemma 4 and inequality (1), we have

$$\begin{aligned} s_v(Q^2) &\leqslant \Delta s_v(Q) + 2s_v(A^2) \\ &\leqslant \Delta s_v(Q) + 6s_v(A) + 2(n - 4) \\ &= \Delta s_v(Q) + 3s_v(Q) + 2(n - 4). \end{aligned}$$

Solving the quadratic inequality, we obtain

$$\lambda_1(Q) \leqslant \frac{\Delta + 3 + \sqrt{(\Delta + 3)^2 + 8(n - 4)}}{2}.$$

By Lemma 1, inequality (2) holds.

If G is a Halin graph, by Lemma 5 and inequality (1), we have

$$\begin{aligned} s_v(Q^2) &\leqslant \varDelta s_v(Q) + 2s_v(A^2) \\ &\leqslant \varDelta s_v(Q) + 4S_v(A) + 2(n-2a+1) \\ &= \varDelta s_v(Q) + 2s_v(Q) + 2(n-2a+1). \end{aligned}$$

Solving the quadratic inequality, we obtain

$$\lambda_1(G) \leqslant \frac{\varDelta + 2 + \sqrt{(\varDelta+2)^2 + 8(n-2a+1)}}{2}.$$

By Lemma 1, inequality (3) holds immediately. □

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